Lower Bounds for Cops and Robber Pursuit

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Abstract. We prove that the robber can evade (that is, stay at least unit distance from) at least $\lfloor n/5.889 \rfloor$ cops in an $n \times n$ continuous square region, that a robber can always evade a single cop in a square with side length 4.5, and that a single cop can always capture the robber in a square with side length smaller than $2.189\cdots$. We also make many new observations and conjectures about both the continuous problem and the discrete problem in which a robber tries to evade k cops on an $n \times n$ grid.

Key words. Pursuit, evasion, cops and robber, lion and man, rabbit and robot, tractrix

AMS(MOS) subject classifications. 49N75

1 Introduction

Under what conditions can a man escape from a pursuing lion? That question has led to the mathematical analyses of a large number of variant problems depending on details such as: Are time and/or space are discrete or continuous? Is space finite or infinite? Does it have obstructions? Are the strategies of the lion and man fixed or adaptive? Are their speeds comparable? Can they move with unrestricted curvatures? Is the position of each known exactly, approximately, or not at all by the other? What does "capture" mean? The literature of pursuit/evasion games is much too broad to be summarized here: Isaacs [13] is a classic, though now dated reference; some historical information can be found in Nahin [21]. We give only a smattering of references, together with details of the results particularly relevant to this paper.

Pursuit problems have been studied at least since the seventeenth century, but the lion-and-man style problems date from a problem posed by Tibor Radó (as given in Littlewood [19, pp. 114–117]): a lion and man move around in the interior of a circle; both move continuously in time and space and each is limited

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to the same speed. Littlewood used a result of Abram Samoilovitch Besicovitch to show that the man can move so as to avoid capture indefinitely. Variations have been studied by Alonso, Goldstein, and Reingold [2], Croft [5], Flynn [7], [8], [9], Gale (see Guy [11]) and Sgall [24], Goldstein and Reingold [10], Isler, Kannan, and Khanna [14], Lewin [18], Merz [20], and Rote [23].

Although the problem is usually stated as a pursuit of a man by a lion (or a rabbit by a robot [12]), the versions we consider are more naturally described as a robber evading cops on patrol. Specifically, we imagine that cops patrol a region on fixed routes and the robber has full knowledge of the cops' routes, but the cops know nothing of the robber's position. In the discrete problem the cops and the robber traverse (at most) one edge of a graph simultaneously with each tick of the clock.[1] In the continuous problem, the cops and the robber have the same maximum speed and move continuously in a continuous region. Dumitrescu, Suzuki, and Zylinski [6] asked, what is the maximum number of cops that a robber can evade on either an $n \times n$ discrete grid or on an $n \times n$ (continuous) square region? Among other results, they proved that $\Omega(\sqrt{n})$ cops can be evaded in either case. Berger, Grüne and Klein [3] improved this result to $\lfloor n/2 \rfloor$ cops in the discrete case, as well as giving a variety of results for higher dimensions.

In this paper we improve the bound of [6] in the continuous case by using the results of [3] with a new discretization lemma to prove that the robber can evade at least $\lfloor n/5.889 \rfloor$ cops in an $n \times n$ continuous square region. We also prove that a robber can always evade a single cop in a square of side length 4.5, and that a single cop can always capture the robber in a square of side length smaller than $2.189\cdots$. Though our focus is on the continuous version of the problem, in the course of our discussion we make a number of observations and conjectures about the discrete problem in which a robber tries to evade k cops on an $n \times n$ grid. Because our proofs rely on the details of the method of [3], we begin with a summary of their lower bound arguments for the two-dimensional discrete case. We follow that summary with our analysis of the continuous case, including a discussion of the power of a single cop. Finally, we conclude with a variety of musings and suggestions for future research.

2 The Discrete Problem

In the discrete version of the cops and robber problem, we are given an undirected graph of vertices and edges. The cops and robber move along an edge from one vertex to another with each tick of the clock; they also have the option of staying in place as the clock ticks. Cops move patrol (move) non-adaptively without knowledge of the robber's position, but the robber always knows where the cops are and where they will move (he's done his homework!). The robber wants to avoid capture by a cop, defined as occurring either when the robber and a cop arrive at the same vertex at the same time, or if the robber and

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[1] The discrete problem considered here differs from that studied by [1], [22], and others because in their formulation the cops move adaptively, not on fixed routes.

a cop traverse the same edge (in opposite directions) at the same time. How many cops need to patrol the graph to guarantee capture of the robber on a Manhattan-like grid of city streets? The grid is the graph G_n which has n^2 vertices in an $n \times n$ array, each vertex connected by an edge to the vertices above, below, left, and right; boundary vertices lack the obvious edges.

To prove their bound on the number of cops, [3] studied the number of neighboring vertices to a set of vertices in G_n; this will tell us that at any time in the pursuit, the robber has a generous number of possible locations that he can reach. By the *neighboring vertices* to a set of vertices S we mean the set of all vertices not in S that have at least one edge connecting them to a vertex in S; we denote the neighbors of S by $N(S)$. They ([3]) use the following lemma

Lemma 1. [3, Lem. 7] *Let S be a subset of vertices in G_n. If*

$$\frac{n(n-1)}{2} < |S| < \frac{n(n+1)}{2},$$

then $|N(S)| \geq n$. $\square$

Lemma 1 follows from a deep and general result of Bollobás and Leader [4, Thm. 8]; [3] gives a direct combinatorial argument for this special case. The lemma guarantees the existence of many safe and accessible vertices for the robber: a vertex is *safe* if neither it nor any neighbor is occupied by a cop; a vertex is *accessible* if it is either occupied by the robber or adjacent to the robber. [3] uses Lemma 1 to prove the following theorem about G_n. Because our results follow from a similar pattern, we give a proof of this theorem.

Theorem 1. [3, Thm 2.] *If there are at most $\lfloor n/2 \rfloor$ cops, a robber can forever evade capture on G_n.*

Proof. We prove that whatever paths the cops patrol, the robber can avoid capture forever. Assume there are k cops, $k \leq \lfloor n/2 \rfloor \leq n^2/2$. Then that at time $t = 0$, there are at least $\lfloor n^2/2 \rfloor$ safe positions for the robber. But Lemma 1 tells us that if at time t there are $\lfloor n^2/2 \rfloor$ safe and accessible vertices for the robber to occupy, then there will be at least n neighboring vertices to which he can move, or he can stay where he is in one of the $\lfloor n^2/2 \rfloor$ vertices, a total of $n + \lfloor n^2/2 \rfloor$ possible moves for the robber at time $t + 1$. Each cop "threatens" only two vertices, his location at time t and his location at time $t + 1$; thus the cops' positions forbid at most $2k$ positions accessible to the robber at time $t+1$. In other words, at least $n + \lfloor n^2/2 \rfloor - 2k$ vertices are safe and accessible for the robber at time $t + 1$. But $k \leq \lfloor n/2 \rfloor$, so at time $t + 1$ there are at least

$$n + \lfloor n^2/2 \rfloor - 2\lfloor n/2 \rfloor \geq \lfloor n^2/2 \rfloor \tag{1}$$

safe and accessible vertices for the robber to occupy at time $t + 1$. Thus for any finite time t and any set of cops' paths of length t, the robber can always find a sequence of t moves from safe position to safe position, each time having at least $\lfloor n^2/2 \rfloor$ outcomes at every level. If we form a decision tree of safe moves for the robber by connecting a root to the accessible positions at time 0, the robber's choices of safe moves thus form an infinite tree (because each tree level t has at least $\lfloor n^2/2 \rfloor$ nodes—safe and accessible positions at time t) and each node except the root has at most 5 children (adjacent positions to the robber's position; the root of tree has $n^2 - k$ starting positions for the robber). Such a tree has an infinite path by the König "infinity lemma" [16] (see [15, sec. 2.3.4.3]); that path gives the infinite sequence of moves for the robber to make to evade the cops forever. □

We make two observations about the proof of this theorem: First the proof remains valid even if the cops can jump great distances! Second, the proof is not constructive—we must give the infinite tree of moves (from time $t = 0$ to ∞) before we can apply the non-constructive "infinity lemma." It would be more satisfying to give a constructive proof in which we need only examine the tree to some depth depending on n and k. This suggests examining a variant problem in which at time t the robber knows only the positions of the cops in the near future, from time t to $t + h(n, k)$ for some horizon function $h(n, k)$.

There is a large gap between the lower bound of Theorem 1 and the obvious upper bound of n cops on G_n, which by marching in a line from one edge to the other will always capture a robber. We believe

Conjecture 1. *If there are fewer than n cops, the robber can forever evade capture on G_n.*

We have accumulated some evidence for this conjecture. Generalizing inequality (1), we see that for k cops, if we have a set of S safe and accessible points at time t, and if

$$|N(S)| + |S| - 2k \geq |S|, \tag{2}$$

we will have at least $|S|$ safe and accessible points at all times after t; this is the gist of the proof of Theorem 1. Define F_n^k to be the minimum number of safe and accessible positions possible for the robber when k cops patrol G_n. Exhaustive computer search gives the values for $n \leq 7$ in Table 1. Together with Lemma 1 and inequality (2), these values prove Conjecture 1 for $n \leq 6$.

Moreover, it is easy to show that

$$F_n^k \leq n^2 - k(k+1)/2. \tag{3}$$

Have k cops sweep from left to right as shown in Figure 1; when the cops reach the right edge, all but $1 + 2 + \cdots + k = k(k+1)/2$ positions are safe and accessible for the robber. This bound is pretty good, as can be seen in the values of $n^2 - k(k+1)/2 - F_n^k$ shown in Table 2: Below the diagonal we have almost all zeroes, meaning perfect agreement between F_n^k and $n^2 - k(k+1)/2$. On the main diagonal the values are so low that Lemma 1 is no longer applicable for the value given in Table 1 for $n = 6$, $k = 5$. But we can improve the bound of (3) on the diagonal: Have the $n - 1$ cops sweep from the left edge of G_n to the right (as in Figure 1). At the right edge they move up one position and

	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$n = 2$	3					
$n = 3$	8	5				
$n = 4$	15	13	8			
$n = 5$	24	22	19	11		
$n = 6$	35	33	30	25	15	
$n = 7$	48	46	43	39	32	≤ 19

Table 1: Values of F_n^k, the minimum of safe and accessible positions for the robber when k cops patrol G_n. Except for F_7^6, all values were found by exhaustive search; inequality (4) gives $F_7^6 \leq 19$.

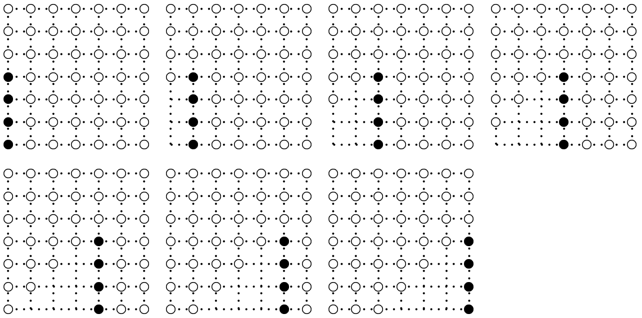

Figure 1: Safe and accessible vertices for the robber as $k = 4$ cops sweep across G_7. Cops are shown as solid dots, safe and accessible positions for the robber are shown as circles. At the end, the remaining $k(k+1)/2$ vertices (at the lower right corner) are either occupied by cops or inaccessible to the robber from a safe and accessible vertex on the previous step.

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
2	0					
3	0	1				
4	0	0	2			
5	0	0	0	4		
6	0	0	0	1	6	
7	0	0	0	0	2	≥ 9

Table 2: Values of $n^2 - k(k+1)/2 - F_n^k$, based on Table 1 and inequality (4).

sweep back to the left edge (see Figure 2). For odd n, when the cops reach the left edge there are

$$n + 2 \times \left(1 + 2 + \cdots + \frac{n-1}{2}\right) = n + \frac{n-1}{2}\frac{n+1}{2} = n + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil ,$$

safe and accessible positions for the robber; for even n there are

$$n + 2 \times \left(1 + 2 + \cdots + \frac{n-2}{2}\right) + \frac{n}{2} = n + \frac{n}{2}\frac{n}{2} = n + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil .$$

Thus,

$$F_n^{n-1} \leq n + \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil . \tag{4}$$

The right hand side of (4) gives the correct value for F_n^{n-1} when $n \leq 6$; does it always? The bound given in inequality (4) is so low that Lemma 1 is not applicable. However, even if $F_n^{n-1} \approx n^2/4$, the border size remains big (which is why we needed Lemma 1 in the first place), giving some small hope for a proof of Conjecture 1 based on inequality (2).

3 The Continuous Problem

The continuous version of the cops and robber problem has both the cops and the robbers moving continuously in an $n \times n$ continuous square region; to capture the robber, a cop must come within (strictly less than) unit distance of him. As in the discrete case, cops move non-adaptively without knowledge of the robber's position, but the robber always knows where the cops are and where they will move. We superimpose a grid on the square region, allowing unrestricted movement by the cops, but limiting the robber's movement to the grid. Next, after proving a "discretization lemma," and reasoning as in Theorem 1, we show that if there are fewer than $\lfloor n/14 \rfloor$ cops, the robber always has an infinite path of safe, accessible vertices on the grid, no matter what the cops' paths are. Then we sharpen the bound by more careful analyses.

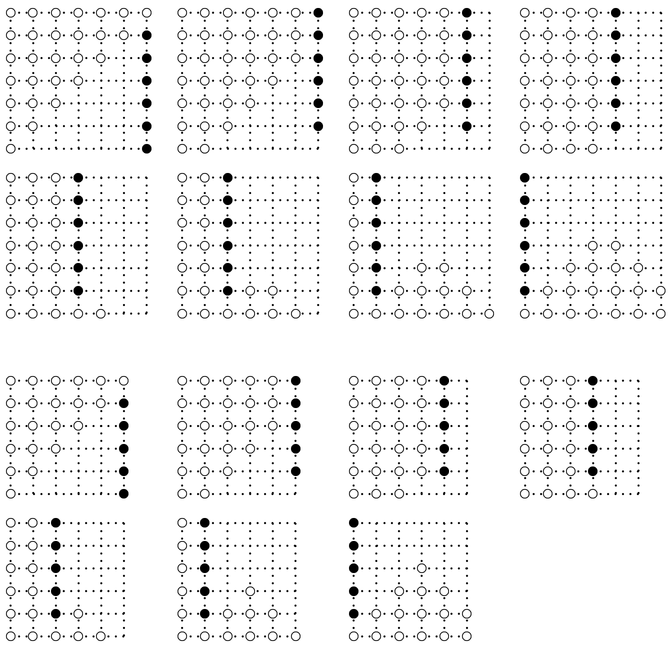

Figure 2: Safe and accessible vertices for the robber as $n-1$ cops sweep from the left edge of to the right edge of G_n, as shown in Figure 1, here continuing: at the right edge they each move up one step and then sweep back from the right edge to the left edge. The upper sequence ($n=7$) shows the case when n is odd; the lower sequence ($n=6$) shows the case when n is even.

Lemma 2. *Superimpose the grid G_n with unit length edges on the $n \times n$ square region with a surrounding border*[2] *of size 0.5, and restrict the robber to moves from vertex to vertex along edges of this grid; cops have unrestricted motion in the square. For a cop to capture the robber during the time interval $[t, t+1]$, as the robber is moving from vertex v_t to vertex v_{t+1}, at time t the cop must be less than 2 units away from v_{t+1}.*

Proof. Without loss of generality, take $t = 0$ and assume that capture occurs at time δ, $0 \leq \delta \leq 1$, as the robber is going from grid position $(0, 0)$ to grid position $(1, 0)$. Let (x, y) be the cop's position at time 0. The situation is illustrated in Figure 3: At time δ the cop must be in the circle of radius δ around his position at time 0. To be captured at position $(\delta, 0)$, the robber must be within distance 1 of the cop, that is, within a circle of radius 1 centered at $(\delta, 0)$. The cop can capture the robber at $(\delta, 0)$ only if the circles overlap, that is, if

$$(x - \delta)^2 + y^2 < (1 + \delta)^2. \tag{5}$$

From time δ until time 1, the cop can move parallel to robber, maintaining a fixed distance from the robber, so that this inequality also holds for time $\delta = 1$. That is

$$(x - 1)^2 + y^2 < 2^2, \tag{6}$$

meaning that at time 0 the cop was within distance 2 of the robber's destination, $(1, 0)$. □

Theorem 2. *If there are fewer than $\lfloor n/14 \rfloor$ cops, the robber can forever evade capture in the continuous $n \times n$ square.*

Proof. We restrict the robber to the n^2 vertices of G_n superimposed on the $n \times n$ square, allowing the cops to move freely. We know from Lemma 2 that if the robber moves to a vertex that is 2 or more units away from any cop, he cannot be captured on that move. We will show how a robber can do this indefinitely. We track a cop's movement as it moves from one 0.5×0.5 square to another.

Suppose that at the beginning of a time step, a cop is in a given 0.5×0.5 square that is not near the boundary of the grid (being near the boundary of the grid makes the numbers even more favorable); there are only 4 surrounding grid vertices on which the robber would be subject to immediate capture and another 11 which could lead to capture on the next move (see Figure 4); Lemma 2 guarantees that any grid vertices further away are safe for the robber, both now and during the next time step. Of these 15 dangerous (to the robber) vertices, a cop cannot simultaneously threaten two that are widely separated (P_1 and P_2 in Figure 4). So each cop can threaten only 14 grid points; that is, k cops can threaten at most $14k$ grid points—cops near the edge of the grid threaten fewer grid points.

We now argue exactly as in the proof of Theorem 1, *mutatis mutandis*: Assume there are k cops, $k \leq \lfloor n/14 \rfloor$ and $14k \leq n^2/2$. Then that at time $t = 0$,

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[2] Because G_n with unit-length edges has dimensions $(n - 1) \times (n - 1)$.

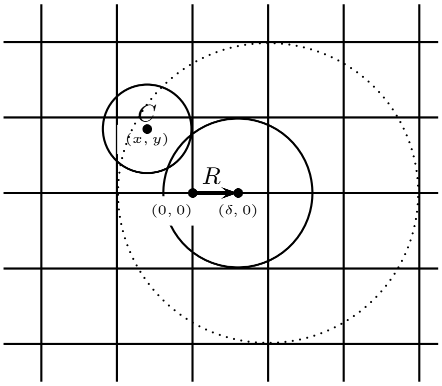

Figure 3: At time δ as the robber R moves from $(0,0)$ to $(1,0)$, the cop C must be in the circle of radius δ around his position at time 0, (x,y). To be captured at $(\delta,0)$, the cop must be within a circle of radius 1 centered at $(\delta,0)$. Thus the cop can capture the robber at $(\delta,0)$ only if the circles overlap, that is, if $(x-\delta)^2+y^2<(1+\delta)^2$; that is, if the cop's initial position is within the dotted circle—a circle of radius 2, centered at $(1,0)$.

there are at least $\lfloor n^2/2\rfloor$ safe grid points for the robber. But Lemma 1 tells us that if at time t there are $\lfloor n^2/2\rfloor$ safe and accessible vertices for the robber to occupy, then there will be at least n neighboring grid points to which he can move, or he can stay where he is in one of the $\lfloor n^2/2\rfloor$ vertices, a total of $n+\lfloor n^2/2\rfloor$ possible moves for the robber at time t. We have seen that each cop "threatens" at most 14 vertices; thus the cops' positions forbid at most $14k$ positions accessible to the robber at time $t+1$. In other words, at least $n+\lfloor n^2/2\rfloor-14k$ grid points are safe and accessible for the robber at time $t+1$. But $k\le\lfloor n/14\rfloor$, so at time $t+1$ there are at least

$$n+\lfloor n^2/2\rfloor-14\lfloor n/14\rfloor\ge\lfloor n^2/2\rfloor$$

safe and accessible vertices for the robber to occupy at time $t+1$. Thus as in Theorem 1, the tree has an infinite path, so the robber can evade the cops forever. □

The bound of Theorem 2 can be sharpened slightly:

Corollary 1. *If there are fewer than $\lfloor n/11\rfloor$ cops, the robber can forever evade capture in the continuous $n\times n$ square.*

Proof. We do a more careful analysis of the vertices prohibited to the robber by a cop, showing that, except for the initial positions, if at each step the robber avoids the 14 threatened positions by a cop, then in the next step this cop prohibits only 11 accessible vertices to the robber. Assume that there are k cops, $k\le\lfloor n/11\rfloor$ and, for the initial position, $14k\le n^2/2$. Then at time

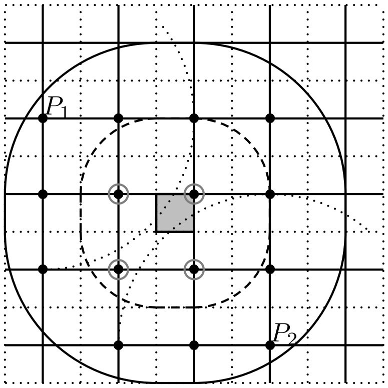

Figure 4: If a cop is in the center gray 0.5×0.5 square, a robber at any of the four circled vertices inside the dashed rounded-square would be subject to immediate capture because they are less than 1 unit away from a possible cop's position. A robber moving to the 15 vertices shown in the solid-line rounded-square could be captured during the next unit-time interval because he is within distance 2 of a possible cop's position, as per Lemma 2. That is, a cop "threatens" at most 15 grid vertices. However, the vertices P_1 and P_2 are so widely separated, $\overline{P_1P_2} = 3\sqrt{2} > 4$, that a cop must choose which of them to threaten by being within distance 2 (the dotted circles of radius 2, centered at P_1 and P_2, do not intersect). Thus each cop threatens only 14 vertices for the robber's present position or his next position.

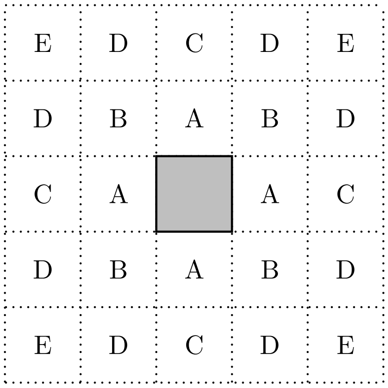

Figure 5: The 24 possible new subgrid locations after unit time by a cop in the gray center subgrid square. There are five different spacial relations between the initial and the final subgrid squares; these are labeled A–E. The cop can also remain in the gray center subgrid square.

$t = 0$, there are at least $\lfloor n^2/2 \rfloor$ safe grid points for the robber. The proof then continues as in Theorem 2, but with "14" changed to "11".

To see why we can change "14" to "11" in the theorem, note that there are 24 0.5×0.5 subgrid squares to which a cop can move in unit time, or he can stay in the same subgrid square, but there are only five different spacial relations between the starting and ending subgrid squares; these spacial relations are shown in Figure 5 with each spacial relation indicated by a letter, A–E. Figure 6 shows relation E, a diagonal move of two subgrid squares.

Three of the vertices, which are prohibited when the cop is in the starting square and accessible in one move only from vertices prohibited by the cop, are also prohibited when the cop is in the ending square. Therefore these vertices are not accessible to a robber when the cop moves, so only 11 of the 14 vertices threatened by a cop in the subgrid square E are accessible to a robber. Cases A–D, and the case when the cop stays in the same subgrid square, are similar, each having at most 11 threatened, accessible vertices. ☐

The ideas used to prove the $\lfloor n/11 \rfloor$ bound can be pushed much further by scaling the superimposed grid and the time interval. First, we generalize Lemma 2.

Lemma 3. *Given an integer $s \geq 1$, superimpose the grid G_{sn} with edge length $1/s$ on the $n \times n$ square region with a surrounding border of size $0.5/s$, and restrict the robber to moves from vertex to vertex along edges of this grid; cops have unrestricted motion in the square. For a cop to capture the robber during the time interval $[t, t + 1/s]$, as the robber is moving from vertex v_t to vertex v_{t+1}, at time t the cop must be less than $1 + 1/s$ units away from v_{t+1}.*

Proof. The argument directly parallels to the proof of Lemma 2. Inequality (6) becomes

$$(x - 1/s)^2 + y^2 < (1 + 1/s)^2. \tag{7}$$

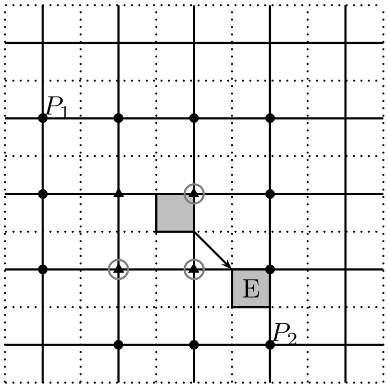

Figure 6: Details of case E from Figure 5. Of the four vertices shown as triangles (which are prohibited and only accessible in one move only from vertices that were also prohibited when the cop was in the initial square), the three circled vertices are also prohibited when the cop moves to E. Therefore these vertices are not accessible to a robber when the cop moves, so only 11 of the 14 vertices threatened by a cop in the subgrid square E *are* accessible to a robber. Recall that P_1 and P_2 cannot be simultaneously threatened by a cop in the initial square.

□

Theorem 3. *If there are fewer than* $\lfloor 10n/69 \rfloor$ *cops, the robber can forever evade capture in the continuous* $n \times n$ *square.*

Proof. Take $s = 10$ in Lemma 3. Figure 7 shows which vertices are threatened by a cop in a subgrid square and the effect of movement orthogonally or diagonally. Thus each cop threatens at most 69 vertices accessible to the robber. Arguing as in the previous theorems, if the number of cops is less the $\lfloor 10n/69 \rfloor$, the robber has an infinite path to avoid the cops. □

Figure 7 used in Theorem 3 is instructive: Of the 352 grid vertices within unit distance of a cop in the center grid square, only 69 are unavailable to the robber (in Corollary 1 it was 11 out of 14). For even better bounds we want to make s larger, but we need an analogue of Figure 7. It is notationally simpler to consider the grid G_n with unit length edges, and ask how many vertices are at distance d of a vertex (ignoring the special cases near the boundaries). That will allow us to compute a bound on the number of vertices accessible to the robber and threatened by a cop in a grid square.

We now give a sequence of lemmas that count the numbers of grid vertices (lattice points) within certain distance properties; although there is a considerable literature on such problems (see, for example, [17]), we have found nothing useful to our context. Let $\mathcal{D}_p^r$ be the grid vertices in an open disk of radius r centered at the point p; that is, $\mathcal{D}_p^r$ consists of all grid vertices at distance less

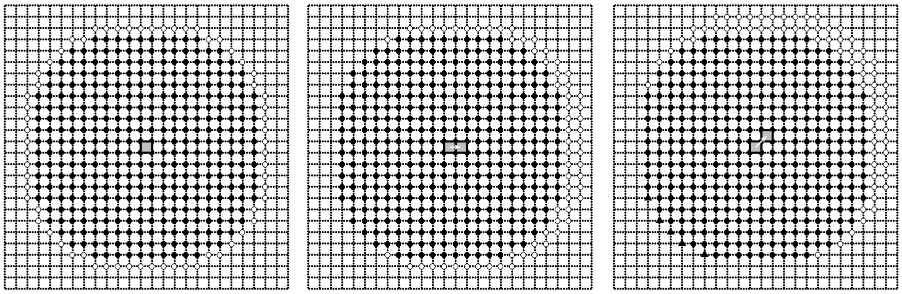

Figure 7: The left figure shows the vertices threatened by a cop in the center (gray) subgrid square when G_{10n} with edge length 0.1 is superimposed on the $n \times n$ region. The 416 vertices within distance 1.1 of the cop are black or white; among them, the 64 white vertices are on the border. The center figure shows what happens when the cop moves one square orthogonally (white arrow), leaving the 64 white vertices accessible to the robber and threatened by the cop. The right figure shows what happens when the cop moves one square diagonally, leaving the 69 white vertices accessible to the robber and threatened by a cop in the center (gray) subgrid square. In this last case, the triangle vertices at the lower left are no longer threatened by the cop.

than r from p (which is not necessarily a grid vertex). More generally, for any set of points S, let $\mathcal{D}_S^r$ be the grid vertices at distance less than r from at least one point S. We partition a set R of grid vertices into *boundary vertices* and *interior vertices*: Boundary vertices $B(R)$ are vertices in R that are connected to at least one vertex outside of R; interior vertices $I(R)$ are vertices in R all of whose neighbors are also in R. For example, in the left figure of Figure 7, if $\boldsymbol{c}$ is the gray center grid square, $I(\mathcal{D}_{\boldsymbol{c}}^{11})$ are the black vertices and $B(\mathcal{D}_{\boldsymbol{c}}^{11})$ are the white vertices.

Lemma 4. *For any grid G_n with unit edge length, any integer r, and any grid vertex p sufficiently far from the boundary of G_n,*

$$|B(\mathcal{D}_p^r)| = \begin{cases} 8\lfloor r\sqrt{2}/2\rfloor + 4 & \textit{if } \lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2, \\ 8\lfloor r\sqrt{2}/2\rfloor & \textit{otherwise.} \end{cases}$$

Proof. With p sufficiently far the edges of G_n, we can count the boundary points per octant (45° sector); if p is close to the edge of G_n, it will have fewer vertices in $|B(\mathcal{D}_p^r)|$. Each octant has $\lfloor r\sqrt{2}/2\rfloor + 1$ boundary vertices (see Figure 8) and all octants are equivalent through rotation and/or reflection. Summing over all eight octants gives us the total number of boundary vertices, but it double-counts the vertices shared by adjacent octants: The vertices at the four cardinal compass points $(\pm(r-1), 0)$, $(0, \pm(r-1))$ are counted twice. The four diagonal compass points $(\pm\lfloor r\sqrt{2}/2\rfloor, \pm\lfloor r\sqrt{2}/2\rfloor)$ may or may not be in $B(\mathcal{D}_p^r)$ because if

$$\lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2 \tag{8}$$

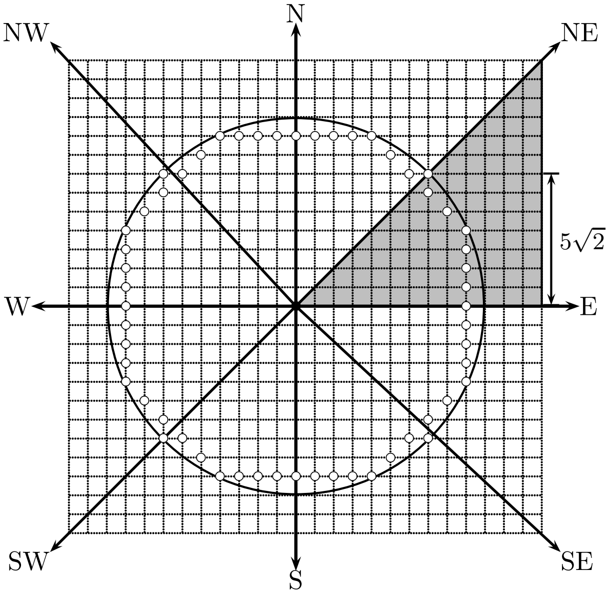

Figure 8: G_{26} with a point p shown in black, a circle of radius $r = 10$, and vertices in $B(\mathcal{D}_p^{10})$ in white. Note that in the first octant $[0^\circ, 45^\circ]$ (shaded gray) there is exactly one boundary vertex per row, for a total of $\lfloor 10\sqrt{2}/2 \rfloor + 1 = 8$ boundary vertices.

(for example, $r = 11$), then the last point on the positive main diagonal, $(\lfloor r\sqrt{2}/2 \rfloor, \lfloor r\sqrt{2}/2 \rfloor)$, has all four neighbors within distance r of p, so it is not a boundary point and hence is not double-counted. On the other hand, if inequality (8) does not hold (for example $r = 10$, shown in Figure 8), the last point on the positive main diagonal has its neighbor above and its neighbor to the right at distance greater than r from p, so it is a boundary point; in this case the last points on the main diagonals are double-counted. $\square$

It is our (necessarily) idiosyncratic definition of boundary points that makes the available literature on lattice points unhelpful. The sequence of numbers of boundary points for a circle of radius at most r, $r = 1, 2, 3, \ldots$, is 4, 8, 16, 20, 28, 32, 36, 44, 48, 56, 60, 64, 72, 76, 84, 88, 96, 100, 104, 112, 116, 124, 128, 132, 140, 144, 152, 156, 164, 168, 172, 180, 184, 192, 196, 200, ... This sequence (divided by 4) is not in *The On-Line Encyclopedia of Integer Sequences* (`http://www.research.att.com/~njas/sequences/`); neither is the sequence of boundary points for a circle of radius *less than* r, 0, 8, 16, 20, 24, 32, 36, 44, 48, 56, 60, 64, 72, 76, 84, 88, 96, 100, 104, 112, 116, 124, 128, 132, 140, 144, 152, 156, 160, 168, ...

We extend Lemma 4 to the boundary vertices distance r from a cell in the grid.

Lemma 5. *For any grid G_n with unit edge length, any integer r, and any grid cell $\boldsymbol{c}$ sufficiently far from the boundary of G_n,*

$$|B(\mathcal{D}^r_{\boldsymbol{c}})| = \begin{cases} 8\lfloor r\sqrt{2}/2\rfloor + 8 & \text{if } \lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2, \\ 8\lfloor r\sqrt{2}/2\rfloor + 4 & \text{otherwise.} \end{cases}$$

Proof. Suppose the cell $\boldsymbol{c}$ is defined by the four corner vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$. Then $B(\mathcal{D}^r_{\boldsymbol{c}})$ is comprised of the boundary vertices of the southwest quadrant of $(0,0)$, the southeast quadrant of $(1,0)$, the northeast quadrant of $(1,1)$, and the northwest quadrant of $(0,1)$. Counting these boundary vertices is the same as in the proof of Lemma 4, except the boundary vertices at the four cardinal compass points $(\pm(r-1),0)$, $(0,\pm(r-1))$ are *not* counted twice. Thus $|B(\mathcal{D}^r_{\boldsymbol{c}})| = |B(\mathcal{D}^r_p)| + 4$. □

If a cop moves orthogonally, say to the right from vertex (i,j) to vertex $(i+1,j)$, the grid vertices that are less than distance r from $(i+1,j)$ and accessible by the robber are the vertices in $\mathcal{D}^r_{p'} - I(\mathcal{D}^r_p)$, corresponding to the white vertices in center part of Figure 7. The next lemma counts these vertices.

Lemma 6. *For any grid G_n with unit edge length, any integer r, and any grid vertices $p = (i,j)$, $p' = (i+1,j)$ sufficiently far from the boundary of G_n,*

$$|\mathcal{D}^r_{p'} - I(\mathcal{D}^r_p)| = \begin{cases} 8\lfloor r\sqrt{2}/2\rfloor + 4 & \text{if } \lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2, \\ 8\lfloor r\sqrt{2}/2\rfloor & \text{otherwise.} \end{cases}$$

Proof. Note that $I(\mathcal{D}^r_p) \subset \mathcal{D}^r_{p'}$ and $|\mathcal{D}^r_p| = |\mathcal{D}^r_{p'}|$, so by elementary set theory,

$$|\mathcal{D}^r_{p'} - I(\mathcal{D}^r_p)| = |\mathcal{D}^r_{p'}| - |I(\mathcal{D}^r_p)| = |\mathcal{D}^r_p| - |I(\mathcal{D}^r_p)| = |\mathcal{D}^r_p - I(\mathcal{D}^r_p)| = |B(\mathcal{D}^r_p)|, \quad (9)$$

and the result follows from Lemma 4. □

Lemma 6 tells us what happens when a cop moves orthogonally from a grid cell to its neighbor on the north, east, south, or west.

Lemma 7. *For any grid G_n with unit edge length, any integer r, and any grid cell $\boldsymbol{c}$ sufficiently far from the boundary of G_n, let $\boldsymbol{c}'$ be a cell orthogonally adjacent to $\boldsymbol{c}$.*

$$|\mathcal{D}^r_{\boldsymbol{c}} - I(\mathcal{D}^r_{\boldsymbol{c}'})| = \begin{cases} 8\lfloor r\sqrt{2}/2\rfloor + 8 & \text{if } \lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2, \\ 8\lfloor r\sqrt{2}/2\rfloor + 4 & \text{otherwise.} \end{cases}$$

Proof. Summing (9) quadrant by quadrant gives us the same result as in Lemma 5. □

If a cop moves diagonally, say to the right from vertex (i,j) to vertex $(i+1,j+1)$, the grid vertices that are less than distance r from $(i+1,j+1)$ and accessible by the robber are the vertices in $\mathcal{D}^r_{p'} - I(\mathcal{D}^r_p)$, corresponding to the white vertices in right part of Figure 7. Counting them is similar to, but more intricate than, the case of orthogonal movement by a cop.

Lemma 8. *For any grid G_n with unit edge length, any integer r, and any grid vertices $p = (i, j)$, $p' = (i + 1, j + 1)$ sufficiently far from the boundary of G_n,*

$$|\mathcal{D}^r_{p'} - I(\mathcal{D}^r_p)| = \begin{cases} 6\lfloor r\sqrt{2}/2\rfloor + 2r + 1 & \textit{if } \lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2, \\ 6\lfloor r\sqrt{2}/2\rfloor + 2r - 2 & \textit{otherwise.} \end{cases}$$

Proof. The vertices $I(\mathcal{D}^r_p) - \mathcal{D}^r_{p'}$ are not accessible to the robber when the cop is at p, but become accessible to the robber (not threatened) when the cop moves to p'. Call this set of vertices T; they are shown as triangles at the lower left in the right part of Figure 7. By elementary set theory,

$$|\mathcal{D}^r_{p'} - I(\mathcal{D}^r_p)| = |\mathcal{D}^r_{p'}| - |I(\mathcal{D}^r_p)| + |T] = |\mathcal{D}^r_p| - |I(\mathcal{D}^r_p)| + |T| = |B(\mathcal{D}^r_p)| + |T|.$$

We compute $|B(\mathcal{D}^r_p)| + |T|$ quadrant by quadrant. Because the cop moves to the northeast, the vertices of T can only be to the southwest of p (we prove this formally below).

Thus in the three quadrants that contain no vertices of T (southeast, northeast, and northwest), we can use the same method as in Lemma 4, giving the number of vertices in $|B(\mathcal{D}^r_p)| + |T|$ in each of these quadrants as $2(\lfloor r\sqrt{2}/2\rfloor + 1)$ (if the last point on the diagonal is not in $|B(\mathcal{D}^r_p)|$) or $2(\lfloor r\sqrt{2}/2\rfloor + 1) - 1$ (if it is). Two vertices are double-counted, $(r - 1, 0)$ and $(0, r - 1)$, so compensating for these we obtain a total of

$$|B(\mathcal{D}^r_p)| + |T| = \begin{cases} 6\lfloor r\sqrt{2}/2\rfloor + 4 & \text{if } \lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2, \\ 6\lfloor r\sqrt{2}/2\rfloor + 1 & \text{otherwise.} \end{cases}$$

in the southeast, northeast, and northwest quadrants.

To compute $|B(\mathcal{D}^r_p)| + |T|$ for the southwest quadrant we must understand where the vertices of T are. We claim that $(x, y) \in T$ if and only if $(x - 1, y), (x, y - 1) \in B(\mathcal{D}^r_p)$ and $(x - 1, y - 1) \notin \mathcal{D}^r_p$. [This claim implies that $(x - 1, y)$ and $(x, y - 1)$ are boundary vertices in the southwest quadrant of $\mathcal{D}^r_p$, proving that the vertices of T occur only in the southwest quadrant relative to p.] Suppose $(x, y) \in T$. This vertex is in $I(\mathcal{D}^r_p)$, so $(x - 1, y)$ and $(x, y - 1)$ are in $\mathcal{D}^r_p$. But (x, y) is not in $\mathcal{D}^r_{p'}$, hence $(x - 1, y - 1)$ is not in $\mathcal{D}^r_p$. This vertex, $(x-1, y-1)$, is a neighbor of $(x-1, y)$ and $(x, y-1)$, so these vertices must be on the boundary of $\mathcal{D}^r_p$, that is, they must be in $B(\mathcal{D}^r_p)$. Conversely, suppose that we have two vertices $(x - 1, y), (x, y - 1) \in B(\mathcal{D}^r_p)$ such that $(x - 1, y - 1) \notin \mathcal{D}^r_p$. Then $(x, y) \in I(\mathcal{D}^r_p)$ and $(x, y) \notin \mathcal{D}^r_{p'}$; this means $(x, y) \in T$.

The claim tells us that in the southwest quadrant the set of vertices of $B(\mathcal{D}^r_p) \cup T$ form a path of steps south and east starting at $(-r + 1, 0)$ and ending at $(0, -r + 1)$. All such paths contain exactly $2(r - 1) + 1$ grid vertices, so $|B(\mathcal{D}^r_p)| + |T| = 2r - 1$ in the southwest quadrant. Adding this number to the number of vertices $|B(\mathcal{D}^r_p)| + |T|$ in the other quadrants, and subtracting 2 for the two double-counted vertices, $(-r + 1, 0)$ and $(0, -r + 1)$, proves the lemma. □

Lemma 8 tells us what happens when a cop moves diagonally from a vertex to its neighbor on the northeast, southeast, southwest, or northwest. We use it to determine what happens when a cop moves from a cell to a diagonally adjacent cell.

Lemma 9. *For any grid G_n with unit edge length, any integer r, and any grid cell $\boldsymbol{c}$ sufficiently far from the boundary of G_n, let $\boldsymbol{c}'$ be a cell diagonally adjacent to $\boldsymbol{c}$.*

$$|\mathcal{D}^r_{\boldsymbol{c}} - I(\mathcal{D}^r_{\boldsymbol{c}'})| = \begin{cases} 6\lfloor r\sqrt{2}/2\rfloor + 2r + 5 & \textit{if } \lfloor r\sqrt{2}/2\rfloor^2 + (\lfloor r\sqrt{2}/2\rfloor + 1)^2 < r^2, \\ 6\lfloor r\sqrt{2}/2\rfloor + 2r + 2 & \textit{otherwise.} \end{cases}$$

Proof. Suppose the cell $\boldsymbol{c}$ is defined by the four corner vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$ and $\boldsymbol{c}'$ is to its northeast, defined by the corner vertices $(1, 1)$, $(2, 1)$, $(2, 2)$, and $(1, 2)$. Adding quadrant by quadrant as is Lemma 8, we find the cardinal compass points $(0, r - 1)$, $(0, r - 2)$, $(r - 2, 0)$, $(r - 1, 0)$ are *not* double counted, so the result is 4 more than in that lemma. ☐

Lemmas 5, 7, and 9 allow us to get an upper bound of the number of newly prohibited vertices as a cop moves: Lemma 5 if he moves within a cell, Lemma 7 if he moves to an orthogonally neighboring cell, and Lemma 9 if he moves to an diagonally neighboring cell. For example, if $r = 11$ (Theorem 3), we have 64 newly prohibited vertices when a cop stays in the same cell (Lemma 5) or moves orthogonally (Lemma 7), and 69 newly prohibited vertices when a cop moves diagonally (Lemma 9). With these three lemmas we can prove our strongest purely analytical result:[3]

Theorem 4. *For each $\epsilon > 0$, if there are fewer than*

$$\left\lfloor \frac{n}{2 + 3\sqrt{2}} - \epsilon \right\rfloor$$

cops, the robber can forever evade capture in the continuous $n \times n$ square.

Proof. Let

$$\delta = \frac{2 + 3\sqrt{2}}{n - \epsilon}\epsilon;$$

that is, δ is the solution to

$$\frac{n}{2 + 3\sqrt{2} + \delta} = \frac{n}{2 + 3\sqrt{2}} - \epsilon.$$

The theorem is equivalent to proving that $\lfloor n/(2 + 3\sqrt{2} + \delta)\rfloor$ is a lower bound. Choose

$$r = \left\lceil \frac{3\sqrt{2} + 7}{\delta} \right\rceil,$$

---

[3] Theorem 5 gives a better bound, but relies on an exhaustive search by computer of cases much to large to be handled by other means.

which is one solution to

$$6\left\lfloor \frac{r\sqrt{2}}{2} \right\rfloor + 2r + 5 < (2 + 3\sqrt{2} + \delta)(r-1),$$

and $s = r - 1$. The same proof as in Theorem 3 gives a lower bound of

$$\left\lfloor \frac{ns}{6\lfloor (s+1)\sqrt{2})/2\rfloor + 2(s+1) + 5} \right\rfloor \geq \left\lfloor \frac{n}{2 + 3\sqrt{2} + \delta} \right\rfloor,$$

proving the theorem. ☐

Because $2 + 3\sqrt{2} \approx 6.24$, Theorem 4 improves somewhat on Theorem 3 and gives us an understanding of what happens when the overlaid grid size approaches 0 (that is, $s \to \infty$). Despite a microscopic grid for the robber, however, our method of tracking a cop's movement allows him to move diagonally a distance of $\sqrt{2}$ units while the robber can move only 1 unit—a decided advantage for the cop! To correct this inequity we use a finer subgrid for the cops than for the robber and get our best result:

Theorem 5. *If there are fewer than* $\lfloor 1000n/5889 \rfloor$ *cops, the robber can forever evade capture in the continuous* $n \times n$ *square.*

Proof. We use the same proof as before, but with a grid of side length 1/1000 for the robber, dividing each grid cell into 20 × 20 sub-grid cells to track the cops' movement. We find, by exhaustive computer search, that the maximum number of new prohibited accessible vertices for the robber is 5889. ☐

Of course, we could let the sub-grid cells on which we track the cops' movement become microscopically small compared to the already microscopically small grid on which we track the robber's movement. To do this, we generalize Theorems 3 and 4. For a disk of radius r on a grid of unit length, define the boundary size

$$\rho_r = \max_{\substack{\text{points } p,\ p' \\ |pp'| \leq 1}} |\mathcal{D}_{p'}^{r+1} - I(\mathcal{D}_p^{r+1})|.$$

Then, as in the proofs of Theorems 3 and 4, but with a cop moving from point to point instead of from cell to cell, we find that for each $\epsilon > 0$, if there are fewer than $\lfloor rn/\rho_r - \epsilon \rfloor$ cops, the robber can forever evade capture in the continuous $n \times n$ square. We conjecture that $\rho_r = r(2\sqrt{2} + 3) + O(1) \approx 5.828r$. If so, it would improve the bound of Theorem 4 to

Conjecture 2. *For each* $\epsilon > 0$, *if there are fewer than* $\lfloor n/(3 + 2\sqrt{2}) - \epsilon \rfloor$ *cops, the robber can forever evade capture in the continuous* $n \times n$ *square.*

This would beat Theorem 5.

Alas, we believe that our theorems, and even this conjecture, are far from establishing the truth of the matter! If we place the $\lceil n/2 \rceil$ cops equally spaced on the border of an $(n-\epsilon) \times (n-\epsilon)$ square so that the distance between adjacent cops is $2-\delta$, $\delta = \epsilon/(\lceil n/2 \rceil + 1)$, and have the cops march in a line to the opposite border, a robber cannot escape capture. We conjecture that this is where the truth lies, and hence

Conjecture 3. *When* $n \geq 3$*, if there are fewer than* $\lceil n/2 \rceil$ *cops, the robber can forever evade capture in the continuous* $n \times n$ *square.*

4 How Powerful Is One Cop?

We now look at the limitations of a single cop. In the discrete case, a single cop is woefully impotent: On G_n, we can have $n^2 - 1$ robbers forever evading capture; this is clearly true for $n = 1$, so suppose that $n \geq 2$. At time $t = 0$, we place a robber on the $n^2 - 1$ vertices of G_n not occupied by the cop. Then, at each time step $t > 0$, if the cop remains still, all the robbers remain still. If the cop moves, all the robbers remain still except the 3 on a square (cycle of length 4) containing the source and the destination of the cop; those 3 robbers move around the square synchronously with the cop.

The question of what a single cop can do is much more interesting in an $n \times n$ continuous region—how large such a region can one cop patrol and be certain to catch the robber? Dumitrescu, Suzuki, and Zylinski [6, Thm. 5] proved that the robber can forever evade a single cop for $n \geq 12$ (this also follows from our Theorem 5). It is trivial to observe, as per the concluding discussion of the previous section, that for $n < 2$ a single cop need only march across the midline of the square to capture the robber. Both of these bounds can be strengthened.

To improve the lower bound for the size of a square on which the robber can evade a single cop, we mimic the method used in the discrete case to verify Conjecture 1 for $n \leq 6$. We need yet another variant of Lemma 2.

Lemma 10. *Given* $s \geq 1$*, superimpose the grid* G_n *with edge length* $1/s$ *on the* $(n-1)/s \times (n-1)/s$ *square region and restrict the robber to moves from vertex to vertex along edges of this grid; cops have unrestricted motion in the square. For a cop to capture the robber during the time interval* $[t, t+1]$*, as the robber is moving from vertex* v_t *to vertex* v_{t+1}*, at time* t *the cop must be less than* $s+1$ *units away from* v_{t+1}*.*

Proof. The argument parallels the proof of Lemma 2 taking $\delta = s$ in inequality (5). □

With this lemma we can use exhaustive search to get results for small n and s. Let $F_{n,s}$ denote the minimum of safe positions for the robber on the grid in Lemma 10 with a single cop. For a given radius $s+1$, if n is too small, there will be no safe locations for the robber, so we can ask for the smallest n for which the robber has safe and accessible positions no matter where the cop is; call this value $n(s)$. Exhaustive search gives us the values shown in Table 3; this gives us

Theorem 6. *If there is only a single cop, the robber can forever evade capture in the continuous* 4.5×4.5 *square.*

	$s=1$	$s=2$	$s=3$	$s=4$
$n(s)$	10	14	17	19
$F_{n(s),s}$	51	95	139	74

Table 3: Number of safe and accessible positions always available to the robber as per Lemma 10 with a single cop.

Proof. From Lemma 10, a robber can forever evade a single cop in a square of size $(n(s)-1)/s$ for all of the values given in Table 3. We have $n(4)=19$, proving the theorem. $\square$

Dumitrescu, Suzuki, and Zylinski [6, Sec. 4.1] give a $\Theta(n^2)$ upper bound and an $n^2/144$ lower bound are given for the number of robbers that can co-exist (not come closer than unit distance to one another) in an $n \times n$ square, all avoiding capture by a single cop. The lower bound is based on their result [6, Thm. 5], mentioned above, that the robber can forever evade a single cop 12×12 square. Hence Theorem 6 gives us a better lower bound for this problem:

Corollary 2. *On $n \times n$ continuous region, at least $n^2/30.25 - O(n)$ robbers can co-exist and forever avoid capture by a single cop.*

Proof. Superimpose the $n \times n$ square with a $\lfloor n/5.5 \rfloor \times \lfloor n/5.5 \rfloor$ grid with sub-squares of size 4.5×4.5; the center to center distance between neighboring sub-square is 5.5. Place a robber in each sub-square and have him move as if the cop were in that 4.5 sub-square—that is, when the cop is outside the sub-square the robber uses the point on the border of the sub-square nearest to the cop to determine he (the robber) will move. By Theorem 6, none of the $\lfloor n/5.5 \rfloor^2 = n^2/30.25 - O(n)$ robbers can ever be captured $\square$

This lower bound seems very weak. We can envision robbers moving on an overlaid grid structure, swapping positions as the cop moves between the cells of the grid, generalizing the argument at the beginning of this section: Put the robbers on a grid with side length $\sqrt{2}$ which allows them to move around a cycle of length four yet remain at least unit distance from each other (they will always be greater than unit distance apart except at the midpoints of the edges where they can be exactly unit distance apart). Such an overlaid grid will be $(\lfloor n/\sqrt{2} \rfloor + 1) \times (\lfloor n/\sqrt{2} \rfloor + 1)$, containing about $n^2/2$ vertices. We place robbers on all but $O(1)$ of them in the neighborhood of the cop; as the cop moves, the robbers near him move around some large cycle of vertices to avoid him. There will be difficulties near the edges of the grid, but this argument suggests

Conjecture 4. *On $n \times n$ continuous region, $n^2/2 - O(1)$ robbers can co-exist and forever avoid capture by a cop.*

We now turn to the upper bound for the size of a square in which a single cop can always capture the robber, we examine patrol strategies for the cop.

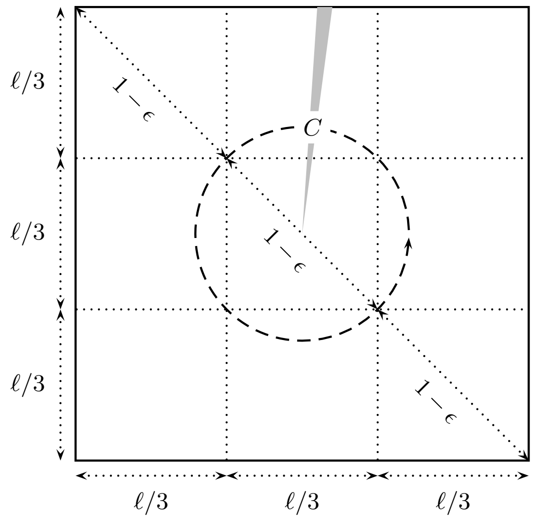

Figure 9: Square with side length $\ell < 3\sqrt{2}/2$. The cop C patrols at full speed counterclockwise around the dashed circle of diameter $1-\epsilon = \ell\sqrt{2}/3$ centered at the center of the square. The robber cannot be in this circle; the robber also cannot be in the gray triangle that sweeps counterclockwise around the center of the square, because that triangle is part of the circle of unit radius around the cop.

Theorem 7. *In the continuous case, a single cop can capture a robber in a square with side length less than* $3\sqrt{2}/2 = 2.121\cdots$.

Proof. In a square of side length $\ell < 3\sqrt{2}/2$ (see Figure 9), the cop patrols at full speed counterclockwise around a circle of diameter $1-\epsilon = \ell\sqrt{2}/3$ centered at center of the square. The robber cannot go into this circle without being captured; the robber must also avoid a sector (shown in gray in Figure 9) that sweeps, like a lighthouse beam, counterclockwise around the center of the square, because that sector is part of the circle of unit radius around the cop. This means that when the cop does a complete circuit, the robber must also make a complete circuit, circumscribing an area larger than that of the circle circumscribed by the cop. Because a circle is the curve of least perimeter enclosing a given area, the robber must be on a longer curve than the cop. But the cop is traveling at top speed, so the cop's "lighthouse beam" sector must overtake the robber, capturing him. $\square$

The proof of Theorem 7 actually proves a stronger result, namely,

Corollary 3. *In the continuous case, a single cop can capture a robber in a circle with diameter less than* 3. □

We conjecture that that this is the strongest possible such result; that is,

Conjecture 5. *In the continuous case, a robber can forever evade a single cop in a circle with diameter* 3.

More generally, what is the minimum diameter circle within which a robber can forever evade k cops?

If Conjecture 5 is correct, Theorem 7 gives, via Corollary 3, a strong bound for a circular region of pursuit; but it does not for a square. The primary aspect of the cop's path in the proof of Theorem 7 is that it goes within unit distance of all points in the square, including the four corners—if it did not, the robber could simply sit on one of the corners to evade capture. In other words, we expect the optimum cop path to be symmetrical, going through some four points, each at slightly less than unit distance from the corner, on the corner bissector. Once these four points are chosen, we expect the path to be a curve of minimum perimeter; this suggests a square trajectory, not a circle, so we now explore what happens when the cop patrols on an axis-aligned square path, centered in the square region.

Let ℓ be the side length of the square and let $\alpha\ell$ be the side length of the cop's square trajectory, axis-aligned and centered in the $\ell \times \ell$ square, as shown in Figure 10. For the cop's path to capture the robber at a corner, we must have the corner of the cop's path less than unit distance from the closest corner of the square region; in other words,

$$\left(\frac{1-\alpha}{2}\ell\right)\sqrt{2} < 1,$$

or

$$\ell < \sqrt{2} + \alpha\ell.$$

To force the robber to travel further than the cop (as in the proof of Theorem 7) we want the arc (shown in dark gray in Figure 10) of the sector of radius 1, centered at the opposite corner of the cop's path and connecting the perpendicular bisectors of the sides of the cop's path opposite the cop's corner position, to be longer than the side of the cop's path. Thus we need

$$\frac{\pi}{2} - 2\theta > \alpha\ell = 2\sin\theta,$$

where θ is the angle between the side of the cop's path and the lower radius of the sector, defined by $\sin\theta = \alpha\ell/2$. If we choose θ so that

$$4\sin\theta = \pi - 4\theta,$$

that is, $\theta = 0.397907756\cdots$ and $\alpha\ell = 2\sin\theta = 0.77498084\cdots$, then

$$\ell < \sqrt{2} + 2\sin\theta = 2.189194376\cdots$$

and $\alpha = 0.35400274\cdots$. We have proved

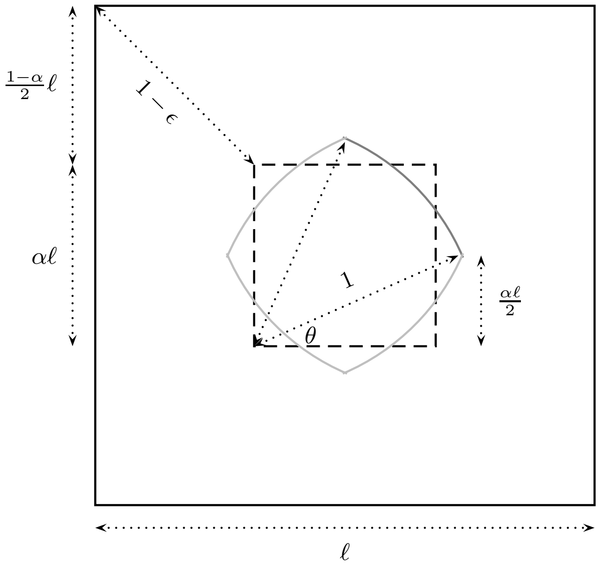

Figure 10: Square with side length $\ell < \sqrt{2} + 2\sin\theta = 2.189194376\cdots$, where $\theta = 0.397907756\cdots$ is the root of $4\sin\theta = \pi - 4\theta$. The cop moves counterclockwise on the axis-aligned centered, internal dashed square with side length $\alpha\ell = 2\sin\theta = 0.77498084\cdots$, $\alpha = 0.35400274\cdots$. The robber must remain outside a region defined by the four gray arcs, one of which is shown in darker gray; the length of each gray arc is $(\pi/2 - 2\theta)$.

Theorem 8. *In the continuous case, a single cop can capture a robber in a square with side length* $\ell < \sqrt{2} + 2\sin\theta = 2.189194376\cdots$*, where* θ *is the root of* $4\sin\theta = \pi - 4\theta$*,* $\theta = 0.397907756\cdots$*.* ☐

We do not believe that Theorem 8 is the strongest possible result, but we believe it is pretty close. We conjecture that the optimum path for a single cop to patrol in a square region is an axis-aligned, centered square path, so we now examine how large the square has to be for the robber to forever evade capture when the cop patrols on a such a path. This examination will lead us to a stronger conjecture.

As the cop patrols along a straight line, how can the robber remain at unit distance from him? Of course, the robber can move parallel to the cop; surprisingly, there is another possibility. Imagine the cop moving at unit speed along the x-axis so that at time t he is at position $(t, 0)$. Let the robber's position at time t be given in parametric form as $(x(t), y(t))$. Because the robber is unit distance from the cop at all times t, we have

$$(t - x(t))^2 + y(t)^2 = 1.$$

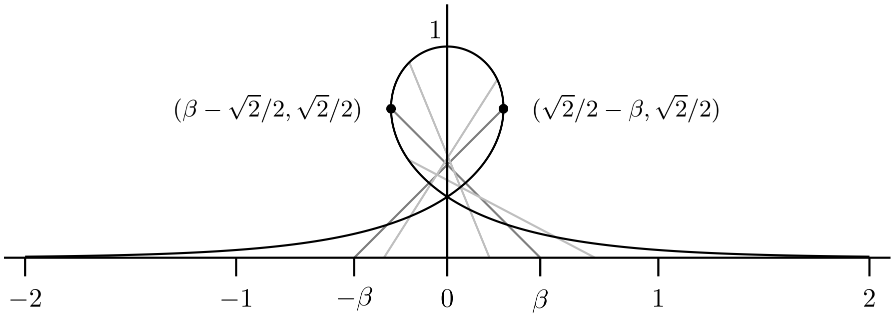

Figure 11: The pseudo-tractrix, defined by parametric equations (10)–(11), is another way for the robber to maintain unit distance from the cop as the cop moves along the horizontal axis. The gray lines show some sample unit distances between the robber and the cop. The unit length darker gray lines from $(-\beta,0)$ to $(\sqrt{2}/2-\beta,\sqrt{2}/2)$ and from $(\beta,0)$ to $(\beta-\sqrt{2}/2,\sqrt{2}/2)$, $\beta=\ln(\sqrt{2}+1)/2=0.44068679\cdots$, make ±45° angles with the x-axis. The "cap" portion of the curve between the points $(\beta-\sqrt{2}/2,\sqrt{2}/2)$ and $(\sqrt{2}/2-\beta,\sqrt{2}/2)$ can be used for the robber's path as the cop goes from $(-\beta,0)$ to $(\beta,0)$.

Furthermore, because the robber (also) moves at unit speed,

$$x'(t)^2+y'(t)^2=1.$$

This system of equations has two solutions: We could have $x'(t)=1$ and $y'(t)=0$, corresponding to the robber moving parallel to the cop. A second, more interesting solution,

$$\begin{aligned} x(t) &= t-\tanh 2t && (10)\\ y(t) &= \operatorname{sech} 2t, && (11) \end{aligned}$$

which we show in Figure 11, could be called a *pseudo-tractrix*. The pseudo-tractrix is a variation on the (linear) tractrix defined by (10)–(11) with the 2s deleted; the tractrix is a pursuit curve studied in the late seventeenth century by Claude Perrault, Isaac Newton, Christian Huygens, Gottfried Wilhelm Leibniz.

Let $\beta=\ln(\sqrt{2}+1)/2=0.44068679\cdots$. We have

$$\begin{aligned} \beta-x(\beta) &= x(-\beta)+\beta=\sqrt{2}/2\\ y(\beta) &= y(-\beta)=\sqrt{2}/2, \end{aligned}$$

so that the angle formed by the horizontal line and the cop-robber line is ±45°. We use the "cap" portion of the pseudo-tractrix, between the points $(\beta-\sqrt{2}/2,\sqrt{2}/2)$ and $(-\beta+\sqrt{2}/2,\sqrt{2}/2)$ shown in Figure 11, to form the robber's path—specifically, as the cop moves on an axis-aligned, centered square path with side length 2β in a square region with side length at least $2(1-\beta)=1.1186264\cdots$ (the minimum size to accommodate the robber's path), a path for the robber that remains at unit distance from the cop can be composed of those portions of four pseudo-tractrices (see Figure 12).

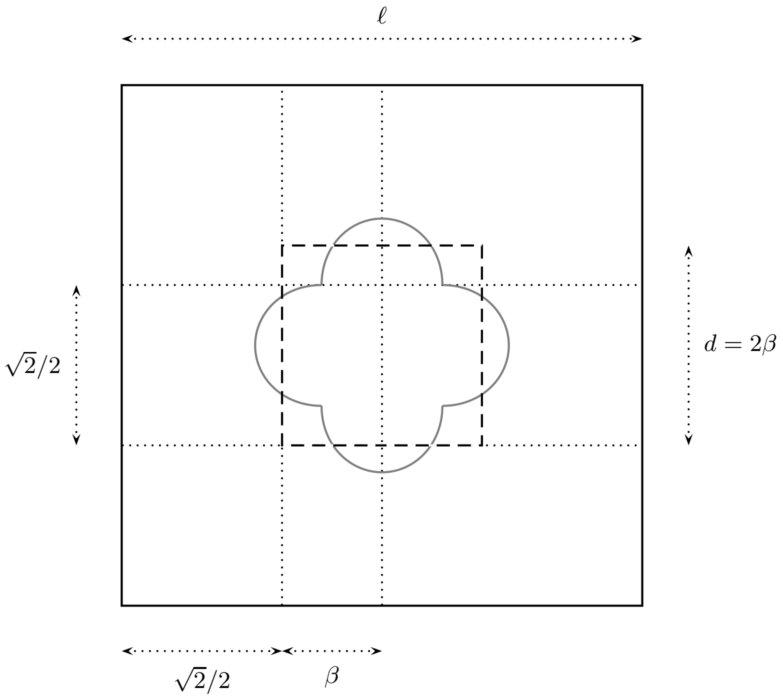

Figure 12: In a square region with side length $\ell = \sqrt{2} + 2\beta$, if the cop moves on the dashed axis-aligned, centered squared path of length $d = 2\beta$, the robber can remain at unit distance from the cop by moving in the same direction as the cop, following the gray pseudo-tractrix arcs on the side opposite the cop. At its widest point the robber's path is $2(1 - \beta)$ units wide.

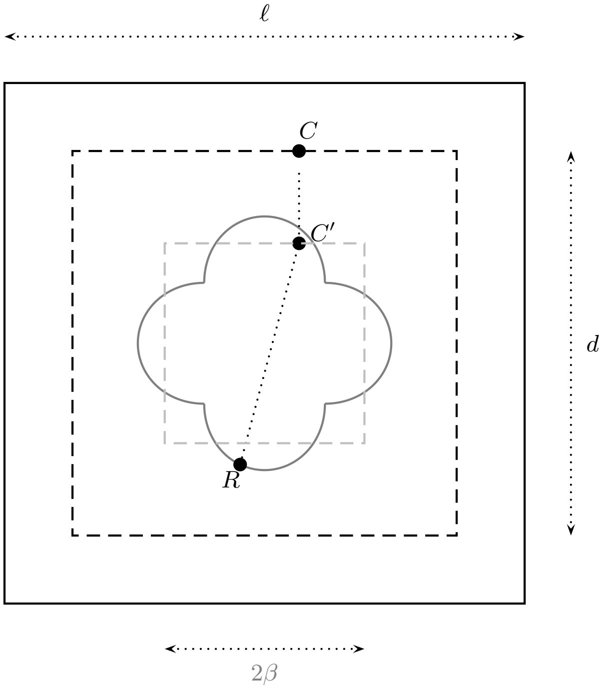

Figure 13: If the cop C moves on an axis-aligned, centered square with side length $d > 2\beta$ (the dashed black line), the robber R moves in the same direction as the cop, following the gray pseudo-tractrix arcs, as though the cop were at C', the closest point to it on the axis-aligned, centered square with side length 2β (the dashed gray line). In this case, the robber moves more slowly than the cop.

Theorem 9. *In the continuous case, if a cop moves on an axis-aligned, centered square path in a square region with side length* $\ell \geq \sqrt{2} + 2\beta = 2.2955\cdots$*, the robber can forever evade the cop.*

Proof. Let d be the side length of the cop's axis-aligned, centered square path. If $d < 2\beta$, then $(\ell - d)/2 > \sqrt{2}/2$ and hence each of the corners of the square region are unit distance or more from the closest point on the cop's path; thus the robber can sit unmoving on a corner without ever being captured. Now suppose that $d = 2\beta$; the robber can move on the "cap" pieces of the pseudo-tractrix as shown in Figure 12, matching the speed of the cop and always staying unit distance from the cop. Finally, if $d > 2\beta$, the robber moves on the pseudo-tractrix arcs as though the cop were moving on the axis-aligned, centered square path with side length $d = 2\beta$, at the position defined by the closest point of this path to the cop's true position (see Figure 13); the robber is always greater than unit distance from the cop and is moving more slowly than the cop. $\square$

Theorem 9 is easily strengthened to *any* square path for the cop:

Corollary 4. *In the continuous case, if a cop moves on any square path in a square region with side length* $\ell \geq \sqrt{2} + 2\beta = 2.2955\cdots$*, the robber can forever evade the cop.*

Proof. Let d be the side length of the cop's square path. If $d < 2\beta$ then, as in the theorem, the furthest corner of the square region from the cop's path must be more than unit distance from the closest point of the cop's path and the robber can sit unmoving at that corner and never be captured. If $d \geq 2\beta$, we propose a possible robber's path, as in the proof of the theorem, based on an imaginary cop's path on a square with side length 2β, centered at the center of the cop's true path and aligned with it. The proposed path is $2(1-\beta)$ units wide at its widest point—that is, it is circumscribed by a circle of radius $1-\beta$ centered at the center of the cop's path. If the proposed path for the robber stays within the square region, the robber can move on this path and avoid the cop. But, if the proposed robber's path goes outside the border of the square region, the center of the cop's true path is at distance less than $1-\beta$ from the border of the square region, hence the opposite border is at least $\sqrt{2} + 2\beta - 2(1-\beta) = \sqrt{2} + 4\beta - 2 > 1$ units away from the closest point of the cop's path. That means that the robber can safely sit unmoving anywhere on that opposite border without being captured. $\square$

The path that we have defined for the robber based on the pseudo-tractrix is not convex, so we can do better if we use a convex path based it instead: A convex path allows the robber to stay at least at unit distance from a smaller cop's path in a smaller region. To determine the convex path, let

$$\gamma = \frac{\sinh^{-1}(\sqrt{2}-1)}{2} = \frac{\ln\left(\sqrt{2}-1+\sqrt{4-2\sqrt{2}}\right)}{2} = 0.20159985958\cdots.$$

We have

$$\begin{aligned}
\gamma - x(\gamma) &= x(-\gamma) + \gamma = \frac{\sqrt{2-\sqrt{2}}}{2} = 0.3826834323\cdots, \\
y(\gamma) &= y(-\gamma) = \frac{1+\sqrt{2}}{2}\sqrt{2-\sqrt{2}} = 0.923879532\cdots, \\
x'(\gamma) &= x'(-\gamma) = y'(\gamma) = -y'(-\gamma) = -\frac{\sqrt{2}}{2},
\end{aligned}$$

so that the angle formed by the horizontal line and the tangent of robber's curve at $\pm\gamma$ is $\pm 135°$. Thus we can use the portion of the pseudo-tractrix between $t = -\gamma$ and $t = \gamma$ to form the top of the robber's path, continuing on the tangent lines on both sides to similar pieces of pseudo-tractrix arc at the sides, and again at the bottom. The resulting path for the robber is shown in Figure 14.

The connecting tangent lines, for example $\overline{R_0R_2}$ in Figure 14, are of length $2g$ with g chosen so that the length of the cop's path as he goes around the

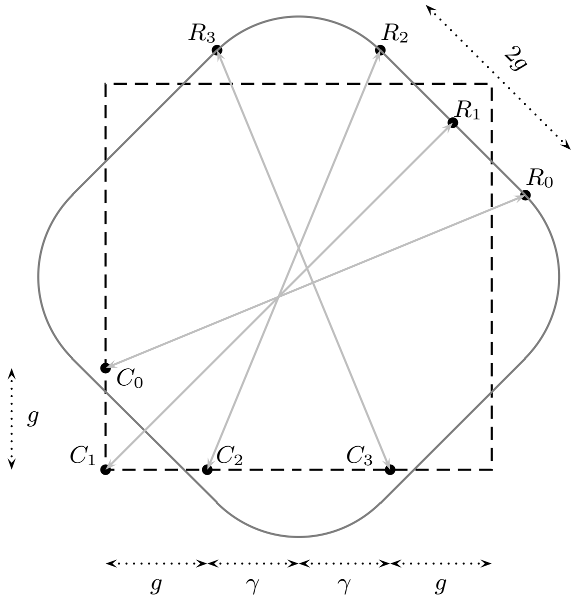

Figure 14: We choose $\gamma = \sinh^{-1}(\sqrt{2}-1)/2$ so that we get a convex path for the robber and $g = (2-\sqrt{2})^{3/2}/2$ so that $\overline{C_0C_1} = \overline{C_1C_2} = \overline{R_0R_1} = \overline{R_1R_2}$. Taking C_1 as the origin, $C_0 = (0, g)$, $C_2 = (g, 0)$, $C_3 = (g + 2\gamma, 0)$, $R_0 = (y(-\gamma), g + \gamma + x(-\gamma))$, $R_1 = (R_0 + R_2)/2$, $R_2 = (g + \gamma + x(-\gamma), y(-\gamma))$, and $R_3 = (g+\gamma-x(\gamma), y(\gamma))$. At its widest point, the robber's path is $2(1-\gamma-g) = 1.14845875167\cdots$ units wide.

corner equals the length of the tangent. In other words, we want g such that

$$g = \overline{R_0R_1} = \overline{R_1R_2} = \overline{C_0C_1} = \overline{C_1C_2}.$$

Taking C_1 as the origin in Figure 14, the coordinates of R_2 are

$$R_2 = (g + \gamma + x(-\gamma), y(-\gamma))$$

and by symmetry the coordinates of R_0 are

$$R_0 = (y(-\gamma), g + \gamma + x(-\gamma)).$$

Because $C_0 = (0, g)$ and $C_2 = (g, 0)$, we want $\overline{R_0R_2} = 2g$, or

$$2\left(y(-\gamma) - x(-\gamma) - \gamma - g\right)^2 = 4g^2.$$

This gives

$$g = \frac{(2-\sqrt{2})^{3/2}}{2} = 0.2241707645839\cdots.$$

Detailed computations show that all cop-robber distances (the light gray lines in Figure 14) are at least unit length. Thus arguments that parallel Theorem 9 and Corollary 4, but using the path in Figure 14, prove

Theorem 10. *In the continuous case, if a cop moves on any square path in a square region with side length $\ell \geq \sqrt{2} + 2g + 2\gamma = 2.265754810702\cdots$, the robber can forever evade the cop.* □

We conjecture that this bound is the best possible; that is,

Conjecture 6. *In the continuous case, a cop can capture the robber if and only if the side length ℓ of the region satisfies*

$$\ell < \sqrt{2} + \left(2-\sqrt{2}\right)^{3/2} + \ln\left(\sqrt{2} - 1 + \sqrt{4-2\sqrt{2}}\right) = 2.265754810702\cdots.$$

5 Conclusions

In our main result we showed (Theorem 5) that in the continuous problem a robber can forever evade $\lfloor n/5.889 \rfloor$ cops on a $n \times n$ square. Unfortunately, this lower bound is far from our conjectured bound of $\lceil n/2 \rceil$ cops (Conjecture 3). There are at least three contributing causes for this gap:

1. Our proof of Theorem 5 is based on Lemma 1—but we know that Lemma 1 leads only to a weak lower bound for the discrete problem: extensive computation shows that Conjecture 1, which is better almost by a factor of 2 than Theorem 1 (which is based on Lemma 1), is true for $n \leq 6$. Moreover, inequality (4) suggests that when n is big enough, Lemma 1 cannot yield an optimal bound.

Similarly, the exhaustive search summarized in Table 3 tells us (with $s = 4$) that a robber can avoid one cop on G_{19}. But using Lemma 1 in conjunction with Lemma 9 (with $r = 5$) we find only that a robber can avoid a cop on G_{30}; again, the weakness lies in Lemma 1. All this suggests that Lemma 1 is too weak and a more powerful lemma will be necessary to obtain optimal bounds.

2. Our analysis lets cops move freely in the sub-square cells, while the robber is restricted to the grid; this has the effect of allowing the cops to move faster than the robber, even if we were to decrease this effect by increasing the number of sub-square cells in Theorem 5.

3. In forcing the robber to move on a grid, we eliminate any possibility of his following general curves—but the results of Section 4 show that general curves may be needed for optimal results. Restricting the robber to a grid is a serious shortcoming of our analysis.

When there is only $k = 1$ cop, the results of Section 4 show that Conjecture 3 no longer applies and the cops' (and the robber's) strategies seem to be very different than for $k > 1$. In Theorem 6, exhaustive computation shows that a robber can forever evade a cop in a square with side length $\ell = 4.5$, while Theorem 8 shows that a cop can always capture a robber on a square with side length $\ell < 2.189\cdots$ by moving on an axis-aligned, centered square path. This latter bound is surely not optimal, but we believe it is close to optimal. Indeed, Theorem 10 shows that a robber can always avoid one cop that uses this strategy on a square with side length $\ell \geq 2.265\cdots$ if the robber uses a convex path formed from pieces of pseudo-tractrices. We conjecture that this bound is the best possible on a square domain: that is, a cop can capture the robber if and only if the side length ℓ of the region satisfies $\ell < 2.265\cdots$. We see no way to extend this strategy to more than one cop.

We saw in Corollary 2 that Theorem 6 gives a better bound on the number of robbers that can co-exist in an $n \times n$ square, all avoiding capture by a single cop. Even proving Conjecture 6 and using it in place of Theorem 6 in Corollary 2 would only improve the bound to about $n^2/10.66$, much weaker than our conjectured $n^2/2 - O(1)$ bound. Different tools are needed to settle Conjecture 4.

For a circular region, we show in Corollary 3 that one cop can always capture a robber on a disk of diameter 3; we conjecture that this is the best possible result. We have found no strategies that show that 2 cops can always capture one robber on a disk with diameter greater than 4. Is that possible?

What happens in higher dimensions? What if the robber and cops move at different speeds? What happens if the robber knows the cops' paths only in the near future—at time t the robber knows only the positions of the cops from time t to $t + h(n)$ for some horizon function $h(n)$? For the $n \times n$ square region, does there exist a finite value $h(n)$ such that there is no difference between our problem and the problem in which the robber only knows the cops' paths t

and $t + h(n)$? In cases in which capture is inevitable, how long can the robber postpone it or how quickly can the cops effect it?

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